Resonance-free regions I

The geometry of trapping, nontrapping estimates in strips, and semiclassical defect measures

Aidan Backus

July 22, 2020

Semiclassical pseudodifferential operators

In what follows we will study "semiclassical" PDE (i.e. those with a small parameter h, "Planck's constant" or the *semiclassical parameter*). Given a linear semiclassical differential operator

$$P(h) = \sum_\alpha a_\alpha(x)(hD)^\alpha$$

we can define its *symbol* $\sum_\alpha a_\alpha(x)\xi^\alpha$ which is a polynomial in ξ. We want an inverse to the map that sends operators to their symbols.

Definition (Definition 4.1.1, "Semiclassical Analysis")

Given $a \in C^\infty(T^*\mathbb{R}^n)$, we define its *standard semiclassical quantization* by its action on $u \in C^\infty_{comp}(\mathbb{R}^n)$:

$$a(x, hD)u(x) = (2\pi h)^{-n} \iint_{T^*\mathbb{R}^n} e^{i\langle x-y,\xi\rangle/h} a(x,\xi)u(y)\; dy\; d\xi.$$

We call $a(hD)$ a *semiclassical pseudodifferential operator* and a its *full symbol*.

Semiclassical pseudodifferential operators

Semiclassical correspondence

Let $a \in C^\infty(T^*\mathbb{R}^n)$. Then we can think of a as a classical observable, so $a(x,\xi)$ describes some property of particles with position x and momentum ξ.
Its quantization is a quantum observable; it acts on wavefunctions u with the property that $\langle a(hD)u(h), u(h)\rangle$ is the expected value of the observable. As $h \to 0$ and $\operatorname{supp} u(h)$ shrinks down to (x,ξ), $\langle a(hD)u(h), u(h)\rangle \to (x,\xi)$, at least in principle.
In particular, while it is not true that $a(hD)b(hD) = (ab)(hD)$, we at least have:

Theorem (correspondence principle; Theorem 4.12, SCA)

Let $a\#b$ be the full symbol of $ab(hD)$ and $\{\cdot,\cdot\}$ the Poisson bracket; then

$$a\#b - ab = \frac{h}{2i}\{a,b\} + O(h^2)$$

in Schwartz seminorms, as $h \to 0$.

The intuition is that $h\{\cdot,\cdot\}$ "looks like a commutator" as $h \to 0$.

Semiclassical pseudodifferential operators

Order of a differential operator

Definition (approximately Definition 4.4.1, SCA)

The *Hörmander symbol class* S^k is defined to consist of those $a \in C^\infty(T^*\mathbb{R}^n)$ such that for every $\ell, j \in \mathbb{N}$,

$$\sup_{(x,\xi)\in T^*\mathbb{R}^n} |\partial_x^\ell \partial_\xi^j a(x,\xi)| \lesssim \langle \xi \rangle^{k-j}.$$

If $a \in S^k$ we write $a(hD) \in \Psi^k$ and say that $a(hD)$ has *order* k.

Theorem (Calderón-Vaillaincourt; Theorem 4.23, SCA)

A pseudodifferential operator is bounded on L^2 iff its order is ≤ 0.

Corollary (Theorem 4.18, SCA)

If $a \in S^k$ and $b \in S^m$ then $a \# b \in S^{k+m}$.

Semiclassical pseudodifferential operators

More useful facts

Theorem (sharp Gårding inequality; Theorem 4.32, SCA)

Assume that $a \in S^0$ *and* $a \geq 0$. *Then if* h *is small enough,*

$$\langle a(hD)u, u\rangle \gtrsim -h||u||_{L^2}^2.$$

Definition (Definition 4.7.1, SCA)

A pseudodifferential operator $a(hD)$ of order m is *elliptic* if one has

$$|a(x, \xi)| \gtrsim |\xi|^m.$$

Corollary (elliptic parametrix construction; Theorem 4.29, SCA)

Elliptic operators are invertible modulo negative-order operators.

Basic setup

Fix n odd, $V \in C^\infty_{comp}(\mathbb{R}^n \to \mathbb{R})$, g a Riemannian metric on $\mathbb{R}^n$ such that $g_{ij} - \delta_{ij}$ has compact support. When working with g we will use Einstein notation.

Definition

The *semiclassical Laplace-Beltrami operator* on $(\mathbb{R}^n, g)$ is the semiclassical pseudodifferential operator $-h^2\Delta_g$ with symbol

$$|\xi|_g^2 = g^{ij}(x)\xi_i\xi_j.$$

Let

$$P(h) = -h^2\Delta_g + V$$

be the semiclassical Schrödinger operator. It follows that $P(h)$ is a semiclassical blackbox Hamiltonian, so the resolvent $R(z,h) = (P(h) - z)^{-1}$ admits a meromorphic continuation to $\mathbb{C}$.
The physical interpretation: "$g_{ij} - \delta_{ij}$ has compact support" means that "gravity is irrelevant near infinity" and "h is small" means that "quantum effects are approximately negligible".

Basic setup

Resonance expansions

Question

When do solutions u of the semiclassical wave equation $-h^2 D_t^2 u = P(h)u$ admit a resonance expansion as $h \to 0$?

Recall that we proved that solutions of potential-scattered wave equations admitted resonance expansions by constructing pole-free regions of the Schrödinger resolvent $R(h)$.

Question

For which half-strips $[\alpha, \beta] \times i[-\nu(h), \infty)$ in $\mathbb{C}$ are there no poles of $R(h)$ in the semiclassical limit $h \to 0$?

Basic setup

Resonance-free regions

Definition

P has a *resonance-free region* of size ν in the energy range $[\alpha, \beta]$ if there are $\delta, h_0 > 0$ such that for every $h < h_0$, every cutoff χ, and every $z \in [\alpha, \beta] \times i[-\nu(h), \infty)$,

$$||\chi R(z,h)\chi||_{L^2 \to L^2} \lesssim_\chi h^{-\delta}.$$

If P has a resonance-free region of size ν then $R(h)$ is holomorphic on $[\alpha, \beta] \times i[-\nu(h), \infty)$ and hence there are no resonances λ with $\lambda^2 \in [\alpha, \beta] \times i[-\nu(h), \infty)$, hence the terminology.

Question

Suppose that P has a resonance-free region of size ν. What is the behavior of $\nu(h)$ in the semiclassical limit $h \to 0$?

Hamiltonian dynamics

Let

$$p(x,\xi) = |\xi|_g^2 + V(x) = g^{ij}(x)\xi_i\xi_j + V(x)$$

be the symbol of P. We introduce the Hamilton vector field

$$H_p = \sum_j \frac{\partial |\xi|^2}{\partial \xi_j}\partial x_j - \frac{\partial V(x)}{\partial x_j}\partial \xi_j$$

which gives a Hamiltonian flow $t \mapsto \exp(tH_p)$,

$$(x(t),\xi(t)) = e^{tH_p}(x(0),\xi(0))$$

on the cotangent bundle $T^*\mathbb{R}^n$. Fix $r_0 > 0$ such that $g_{ij}(x) - \delta_{ij}(x) = V(x) = 0$ if $|x| > r_0$.

Hamiltonian dynamics

The trapped set

Recall that p is the symbol of a Hamiltonian. So if $(x, \xi) \in T^*\mathbb{R}^n$ we view $p(x, \xi)$ as the (classical) energy of a particle at position x and momentum ξ. The energy is invariant along any trajectory of H_p.

Definition

Let $(x(t), \xi(t)) = e^{tH_p}(x(0), \xi(0))$ be a trajectory of H_p. We say that (x, ξ) escapes at time $\pm\infty$ if $|x(t)| \to \infty$ as $t \to \pm\infty$. The tail $\Gamma^{\mp}$ is the set of trajectories that do not escape at time $\pm\infty$.
We say that a trajectory (x, ξ) is *trapped* if $(x, \xi) \in K = \Gamma^+ \cap \Gamma^-$.
We write $\Gamma^{\pm}_J = \Gamma^{\pm} \cap p^{-1}(J)$ and $K_J = \Gamma^+_J \cap \Gamma^-_J$ for the set of trapped trajectories in an energy range J. Here J can be a real number or a set of real numbers.

Hamiltonian dynamics

Physical interpretation

Suppose that $u(h) = u(h, 0)$ is a wave packet that is microlocalized to (x, ξ), in the sense that $||u(h)||_{L^2} = 1$ and there are symbols $\chi(h)$ such that $\operatorname{supp} \chi(h) \subset T^*\mathbb{R}^n$ shrinks down to (x, ξ) and

$$(1 - \chi(hD, h))u(h) = O(h^\infty).$$

Thus u represents a particle that classically has position x and momentum ξ. The dynamics of u are given by the time-dependent semiclassical Schrödinger equation:

$$ih\partial_t u(h, t) = P(h)u(h, t).$$

If h is small enough, then $u(h)$ stays microlocalized to (x, ξ) as $u(h)$ evolves according to the Schrödinger equation and (x, ξ) evolves according to H_p. In particular, (x, ξ) is trapped iff $u(h)$ is – and if $u(h)$ is trapped we might not be able to give $u(h)$ a resonance expansion.

Topology of trapped sets

Theorem

> $\Gamma^\pm$ *is a closed set, and if* $J \subset \mathbb{R}$ *is compact then* K_J *is compact.*
>
> $K_{\mathbb{R}\setminus 0} \subseteq \{(x,\xi) : r(x,\xi) < r_0\}$.
>
> *If* $E \in \mathbb{R}$ *and* $K_E = \emptyset$*, then there is a* $\delta > 0$ *such that* $K_{[E-\delta,E+\delta]} = \emptyset$.

Lemma (escape criteria)

Let (x,ξ) *be a trajectory of* H_p.

> *If* $r(x(0),\xi(0)) \geq r_0$, $\xi(0) \neq 0$*, and* $\pm H_p r(x(0),\xi(0)) \geq 0$ *then* $(x,\xi) \notin \Gamma^\mp$. *Moreover, if* $\pm t > 0$*, then* $r(x(t),\xi(t)) > r_0$ *and* $\pm H_p r(x(t),\xi(t)) \geq 0$.
>
> *If* $(x(0),\xi(0)) \notin \Gamma^\mp$*, then for every* $\pm t$ *large enough,* $r(x(t),\xi(t)) > r_0$ *and* $\pm H_p r(x(t),\xi(t)) \geq 0$.

Thus if $r(x,\xi) > r_0$ and $\pm H_p r(x,\xi) \geq 0$ we can view (x,ξ) as having escaped to infinity.

Recall from Hamiltonian dynamics that $H_p^j r = \partial_t^j r$.
If $r > r_0$ then $p(x,\xi) = |\xi|_\delta^2 = \xi^i \xi_j$. Therefore

$$H_p r(x,\xi) = 2\frac{\xi^i x_i}{r(x,\xi)}$$
$$H_p^2 r(x,\xi) = 4\frac{(x^i x_i)(\xi^j \xi_j) - (\xi^k x_k)^2}{r(x,\xi)^3}.$$

In particular $H_p^2 r \geq 0$ as long as $\xi \neq 0$, so $H_p r$ is increasing as $t \to \infty$.
But $\dot{x} = 2\xi$ and $\dot{\xi} = 0$. So if $H_p r(x,\xi) \geq 0$, it follows that $(x,\xi) \notin \Gamma^{\mp}$.
Conversely, if (x,ξ) is not a trapped trajectory, then clearly $r(x,\xi) > r_0$ eventually and eventually $H_p r \geq 0$. This proves the lemma.

Lemma

Γ^- *is closed.*

To prove the lemma, suppose $(x, \xi) \notin \Gamma^-$. Then there is a $T \geq 0$ such that $H_p r(x(T), \xi(T)) > 0$ and $r(x(T), \xi(T)) > r_0$. These are clearly open conditions so if we perturb (x, ξ) this remains true.

But then the converse to escape criteria implies that the perturbation is also $\notin \Gamma^-$, which implies that Γ^- is the complement of an open set, proving this lemma.

In particular, Γ^+ and hence K is closed.

Topology of trapped sets

Proof of theorem

Lemma

$K_{\mathbb{R}\setminus 0} \subseteq \{(x,\xi) : r(x,\xi) < r_0\}$.

Suppose that (x,ξ) satisfies $\xi \neq 0$ and $r(x,\xi) \geq r_0$. (The condition $\xi \neq 0$ is equivalent to $p(x,\xi) \neq 0$ since $p(x,\xi) = |\xi|_\delta^2$ if $r(x,\xi) \geq r_0$.)
If $H_p r \geq 0$ then $(x,\xi) \notin \Gamma^-$ by the first lemma. Otherwise $H_p r < 0$ so $(x,\xi) \notin \Gamma^+$.
Either way, $(x,\xi) \notin K$. This proves the lemma.
As a consequence, if $J \subset \mathbb{R} \setminus 0$, every $(x,\xi) \in K_J$ satisfies $r(x,\xi) < r_0$. This implies that K_J is compact.

To finish the proof of the theorem we just have to show:

Lemma

If $E \in \mathbb{R}$ and $K_E = \emptyset$ then there is a $\delta > 0$ such that $K_{[E-\delta, E+\delta]} = \emptyset$.

Suppose that there are $E_j \to E$ and $(x_j, \xi_j) \in K_{E_j}$.
If $p(x, \xi) \le 0$ then $0 \le |\xi|_\delta^2 \le -V(x) = 0$ if $r(x, \xi) > r_0$, so (x, ξ) is trapped.
Therefore $E > 0$, so there is a compact $J \subset (0, \infty)$ such that $E, E_j \in J$.
By the previous lemma, K_J is compact, so (after choosing a subsequence if necessary) we may assume that there is a limit $(x_\infty, \xi_\infty) \in K_J$ of the (x_j, ξ_j).
Then $p(x_\infty, \xi_\infty) = E$.
So $(x_\infty, \xi_\infty) \in K_E$, proving the contrapositive of the lemma.

Convergence to trapped sets

Theorem (convergence to trapped sets)

Suppose that $(x, \xi) \in \Gamma_E^\pm$*. Then* $(x, \xi) \to K_E$ *as* $t \to \mp\infty$*. The rate of convergence is uniform in compact subsets of* $\Gamma_E^\pm$*.*

Corollary

If $K_E = \emptyset$ *then* $\Gamma_E^\pm = \emptyset$*.*

It suffices to prove the theorem for Γ_E^- by symmetry; since nonpositive-energy curves are already trapped, we may assume $E > 0$.

Lemma (compactness)

Let $(x, \xi) \in \Gamma_E^\pm$*,* $\rho(t) = r(x(t), \xi(t))$*. Then for every* $t \geq 0$*,*

$$\rho(t) \leq \max(r_0, \rho(0)).$$

Convergence to trapped sets

Proof of compactness

Lemma (compactness)

Let $(x,\xi) \in \Gamma_E^\pm$, $\rho(t) = r(x(t), \xi(t))$. *Then for every* $t \geq 0$,

$$\rho(t) \leq \max(r_0, \rho(0)).$$

The lemma is obviously true for $t = 0$. So if the lemma is false in general, then there is a $T > 0$ such that $\rho(T) > r_0$ and $\rho(T) > \rho(0)$.
Since ρ is continuous and $[0, T]$ is compact, let $t_0 \in [0, T]$ maximize ρ. Then $t_0 > 0$, $\rho(t_0) > r_0$, yet

$$H_p r(x(t_0), \xi(t_0)) = \dot\rho(t_0) = 0.$$

The trapping criteria give $(x, \xi) \notin \Gamma^-$ since $H_p r \geq 0$, a contradiction. This proves the lemma.

Theorem

Suppose that $(x, \xi) \in \Gamma_E^-$. *Then* $(x, \xi) \to K_E$ *as* $t \to \mp\infty$.

Suppose the theorem fails. Then there are $t_j \to \infty$ and a neighborhood U of K_E such that $(x(t_j), \xi(t_j)) \notin U$.
By the compactness lemma, the trajectory (x, ξ) is bounded (and contained in the closed set Γ_E^-), so we may choose a limit point $(x_\infty, \xi_\infty) \in \Gamma_E^-$ of $(x(t_j), \xi(t_j))_j$.
Then $(x_\infty, \xi_\infty) \notin K_E$, hence $(x_\infty, \xi_\infty) \notin \Gamma^+$.
Therefore

$$\lim_{t \to -\infty} r(x_\infty(t), \xi_\infty(t)) = \infty.$$

Let T be so large that $r(x_\infty(T), \xi_\infty(T)) > \max(r_0, r(x(0), \xi(0)))$.

Convergence to trapped sets

Proof of theorem

By continuity of the flow,

$$(x(t_j - T), \xi(t_j - T)) \to (x_\infty(T), \xi_\infty(T)).$$

But T was chosen so large that

$$r(x_\infty(T), \xi_\infty(T)) > \max(r_0, r(x(0), \xi(0))).$$

Thus we can find j so large that

$$r(x(t_j - T), \xi(t_j - T)) > \max(r_0, r(x(0), \xi(0))).$$

This is a contradiction of the compactness lemma which proves the theorem.

Liouville measures

Definition

Let (X, ω) be a symplectic manifold. Let dm be the normalized top exterior power $dm = \omega^{\wedge n}/n!$ of ω. We call m the *canonical measure* on X.

Definition

Suppose that $E \in \mathbb{R}$ is an energy and $dp|_{p^{-1}(E)} \neq 0$. Then we say that $p^{-1}(E)$ is a *nondegenerate energy hypersurface*.

Definition

Let m be the canonical measure on $(T^*\mathbb{R}^n, d\xi \wedge dx)$. If $p^{-1}(E)$ is a nondegenerate energy hypersurface, define a form $\mathcal{L}_E$ by

$$dp \wedge d\mathcal{L}_E = dm.$$

We call $\mathcal{L}_E$ the *Liouville measure* associated to E.

Poincaré recurrence

Theorem

Let m be the canonical measure on $T^*\mathbb{R}^n$; then $m(\Gamma^\pm \setminus K) = 0$. Similarly, if $p^{-1}(E)$ is a nondegenerate energy hypersurface, then

$$\mathcal{L}_E(\Gamma_E^\pm \setminus K_E) = 0.$$

We can just prove this for $p^{-1}(E)$ because the same proof will work for $T^*\mathbb{R}^n$, and similarly we may just prove this for Γ_E^-. There is nothing to prove unless $E > 0$. Since H_p preserves $d\xi \wedge dx$, in particular H_p preserves $\mathcal{L}_E$. Moreover, the Poincaré recurrence theorem says that for an invariant Radon measure, almost every trajectory in a compact set returns to arbitrarily small balls about its initial state infinitely many times.

So these two results, together with the stated theorem and the fact that K_E is compact, guarantee that $\mathcal{L}_E$-almost every trajectory in $\Gamma_E^\pm$ returns to its approximate initial state infinitely many times.

Poincaré recurrence

Proof of theorem

By the compactness lemma, the flow H_p carries $\Gamma_E^- \cap \{r \leq r_0\}$ into itself. Let

$$A_j = e^{tH_p}(\Gamma_E^- \cap \{r \leq r_0\})$$

be the image of $\Gamma_E^- \cap \{r \leq r_0\})$ under H_p at time $j \in \mathbb{Z}$. Then $A_{j+1} \subseteq A_j$.
We already proved that Γ_E^- converges to K_E. Therefore $\bigcap_j A_j = K_E$ and $\bigcup_j A_j = \Gamma_E^-$.
Since A_j is compact, continuity of measure implies that

$$\mathcal{L}_E(K_E) = \lim_{j \to +\infty} \mathcal{L}_E(A_j)$$
$$\mathcal{L}_E(\Gamma_E^-) = \lim_{j \to -\infty} \mathcal{L}_E(A_j).$$

But $\mathcal{L}_E$ is invariant under H_p, so $\mathcal{L}_E(\Gamma_E^-) = \mathcal{L}_E(K_E)$. This proves the theorem.

Resonances in strips

We want to show that given α, β, C, for every h small enough, $[\alpha, \beta] \times i[-Ch, Ch]$ has no resonances $z = \lambda^2$.
Let $P(h)$ be a semiclassical black box Hamiltonian on (M, g); then if h is small enough, the resolvent $R(h)$ meromorphically continues to $[\alpha, \beta] \times i[-Ch, Ch]$.

Definition

Let z be a pole of $R(h)$ and let

$$R(w, h) = \sum_{j=1}^{J} \frac{B_j}{(w-z)^j} + B_z(w)$$

be the Laurent expansion of $R(h)$ at z.
A *resonant state* of $P(h)$ is an element of the image of B_J.

The space of smooth resonant states is finite-dimensional, and if $u(h)$ is a resonant state then $P(h)u(h) = zu(h)$.

Complex scaling

Fix $\theta \in (0, \pi/2)$ and $r_1 > r_0$, and F_θ a smooth, convex function on $\mathbb{R}^n$ with $F_\theta = 0$ on $B(0, r_1)$ and

$$F_\theta(x) = \tan\theta |x|^2/2$$

on $B(0, 2r_1)^c$. Let

$$f_\theta(x) = x + i\partial_x F_\theta(x)$$

and $\Gamma_\theta = f_\theta(\mathbb{R}^n)$ be the usual totally real submanifold.
Let Δ_θ be the restriction of Δ (viewed as an holomorphic differential operator) to Γ_θ. Introduce the complex-scaled operator $P_\theta(h)$ defined by $P_\theta = P$ on $B(0, r_1)$ and $P_\theta(h) = -h^2\Delta_\theta$ on $B(0, r_0)^c$. Then the resolvent

$$(P_\theta - z)^{-1} : L^2(\Gamma_\theta) \to H^2(\Gamma_\theta)$$

is a meromorphic family of operators, and P_θ is a pseudodifferential operator.

Complex scaling

Fiber-radial compactification

Let (M, g) be a Riemannian manifold.

Definition

The *coball bundle* of (M, g) is the fiber bundle

$$B^*M = \{(x, \xi) \in T^*M : g^{ij}(x)\xi_i\xi_j \leq 1\}.$$

One has an open dense embedding $T^*M \to B^*M$ by

$$(x, \xi) \mapsto \left(x, \frac{\xi}{1 + \langle \xi \rangle}\right).$$

Definition

Viewing $B^*_x M$ as a compactification of $T^*_x M$, we call B^*M the *fiber-radial compactification* $\overline{T}^*M$ of T^*M.

Complex scaling

Principal symbols

Let M be a smooth manifold, $\Psi^k(M)$ the space of kth-order semiclassical pseudodifferential operators on M. Let $S^k(M)$ be the space of kth-order symbols on M, and $hS^k(M)$ those symbols which are $O(h)$ as $h \to 0$.

Lemma (Theorem 14.1, SCA)

There is a unique morphism of algebras

$$\sigma_h : \frac{\Psi^k(M)}{\Psi^{k-1}(M)} \to \frac{S^k(M)}{hS^{k-1}(M)}$$

which is the left inverse of the quantization map $a \mapsto a(hD)$ modulo $hS^{k-1}(M)$.

Definition

For every $Q \in \Psi(M)$, $\sigma_h(Q)$ is called the *principal symbol* of Q.

The symbol of a pseudodifferential operator depends on a choice of coordinates, but not the principal symbol.

Complex scaling

The complex-scaled symbol

Lemma (Lemma 6.8, Dyatlov-Zworski)

Let $p_\theta = \sigma_h(P_\theta)$*, and* p *the symbol of* P*. Then:*

- $\operatorname{Im} p_\theta \leq 0$.
- *For every* $E \in \mathbb{R}$, $\{\langle\xi\rangle^{-2}(p_\theta - E) = 0\} \subseteq p^{-1}(E)$.
- *For every* $0 < \alpha \leq \beta$ *there is a* $\delta > 0$ *such that for every* $E \in [\alpha, \beta]$ *and* $x \notin B(0, r_1)^c$,
$$|p_\theta(x, \xi) - E| \geq \delta\langle\xi\rangle^2.$$
- *Fix* $x, \xi, t_0 \leq t_1$*, and consider the flow on* $\overline{T}^*\mathbb{R}^n$,
$$\varphi_t = \exp(t\langle\xi\rangle^{-1} H_{\operatorname{Re} p_\theta}).$$
If for every $t \in [t_0, t_1]$, $\varphi_t(x, \xi) \in \{\langle\xi\rangle^{-2} \operatorname{Im} p_\theta = 0\}$ *then for every* $t \in [t_0, t_1]$,
$$\varphi_t(x, \xi) = \exp(t\langle\xi\rangle^{-1} H_p)(x, \xi).$$

We omit the proofs.

Outgoing estimates

Wavefront sets

Definition

Let $a(h)$ be a symbol. The *essential support* $\operatorname{ess\,supp} a$ of a is the intersection of all compact sets K such that for every symbol $\chi \in S^0$ and every Schwartz seminorm $||\cdot||_{\alpha,\beta}$, if $\chi = 0$ on K, then

$$||\chi a(h)||_{\alpha,\beta} = O(h^\infty),$$

if such a compact set exists. If $\operatorname{ess\,supp} a$ exists, we say that a has *compact essential support*.

The operators $a(hD)$, where a has compact essential support, are exactly those for which there is a compactly supported symbol χ such that the operator seminorms $\mathcal{S}' \to \mathcal{S}$ of $(1 - \chi(hD))a(hD)$ are $O(h^\infty)$.

Definition

The *semiclassical wavefront set* $\mathrm{WF}_h(a(hD))$ of a pseudodifferential operator $a(hD)$ is defined by $\mathrm{WF}_h(a(hD)) = \operatorname{ess\,supp} a$.

Outgoing estimates

Lemma (Proposition 6.9, D-Z)

Let $0 < \alpha \leq \beta$, $C_0 > 0$, $K = [\alpha, \beta] \times i[-C_0 h, C_0 h]$. *Let* $z \in K$, $u \in L^2(\mathbb{R}^n)$. *Let* $f = (P_\theta - z)u$. *Then, with constants independent of* u, z, h*:*

For every pseudodifferential operator A *with compact support and* $\mathrm{WF}_h(A) \cap \Gamma^+_{[\alpha,\beta]} = \emptyset$,

$$||Au||_{L^2} \lesssim h^{-1}||f||_{L^2} + h^\infty ||u||_{L^2}.$$

For every pseudodifferential operator B *with compact support which is elliptic in a neighborhood of* $K_{[\alpha,\beta]}$ *and* h *sufficiently small,*

$$||u||_{L^2} \lesssim ||Bu||_{L^2} + h^{-1}||f||_{L^2}.$$

We omit the proof, which uses the previous lemma, elliptic regularity, propagation of singularities, and the parametrix construction for elliptic operators.

Nontrapping implies resonance-free regions

Definition (Definition 7.1.1, SCA)

The *semiclassical Sobolev norm* of a Schwartz function u is

$$||u||_{H_h^s}^2 = \sum_{|\alpha| \leq s} ||(hD)^\alpha u||_{L^2}.$$

Note that this is just a rescaled version of the Sobolev norm.

Theorem (nontrapping estimate)

Suppose that $[\alpha, \beta] \subset (0, \infty)$, $C_0 > 0$ χ *a cutoff, and* $K_{[\alpha,\beta]} = \emptyset$. *Then for every* $s \geq 0$, $h > 0$ *small, and* $z \in [\alpha, \beta] \times i[-C_0 h, C_0 h]$,

$$||(P_\theta - z)^{-1}||_{H_h^s \to H_h^{s+2}} \lesssim h^{-1}$$
$$||\chi R(z,h)\chi||_{H_h^s \to H_h^{s+2}} \lesssim h^{-1}.$$

Nontrapping implies resonance-free regions

Elliptic parametrix estimates

We must show that for every $f \in C^\infty_{comp}(\mathbb{R}^n)$, $u = (P_\theta - z)^{-1} f$, that

$$||u||_{H_h^{s+2}} \lesssim h^{-1} ||f||_{H_h^s}.$$

By complex scaling, $P_\theta - z$ is elliptic near momentum infinity; that is, if $|\xi| \gg 1$, then

$$|p_\theta(x, \xi) - \operatorname{Re} z| \gtrsim |\xi|^2.$$

Let χ be a cutoff such that $(1 - \chi(hD))(P_\theta - z)$ is elliptic; then there is a parametrix T of $(1 - \chi(hD))(P_\theta - z)$; i.e. T is an inverse of $(1 - \chi(hD))(P_\theta - z)$ modulo terms of order $-\infty$. So

$$||(1 - \chi(hD))u||_{H_h^{s+2}} \lesssim ||f||_{H_h^s} + h^\infty ||u||_{L^2}.$$

Nontrapping implies resonance-free regions

Semiclassical Sobolev estimates

Since $\chi(hD)$ is a frequency cutoff, $||\chi(hD)u||_{H_h^t} \lesssim ||u||_{L^2}$ for any $t > 0$; in particular, the estimate

$$||(1-\chi(hD))u||_{H_h^{s+2}} \lesssim ||f||_{H_h^s} + h^\infty ||u||_{L^2}$$

implies

$$||u||_{H_h^{s+2}} \lesssim ||f||_{H_h^s} + ||u||_{L^2}.$$

On the other hand, the previous lemma said that if $K_{[\alpha,\beta]} = \emptyset$ then for any pseudodifferential operator B of compact support and h small,

$$||u||_{L^2} \lesssim ||Bu||_{L^2} + h^{-1}||f||_{L^2}.$$

In particular this works if $B = 0$, so

$$||u||_{H_h^{s+2}} \lesssim h^{-1}||f||_{L^2}$$

which was to be shown.

Nontrapping implies resonance-free regions

Cutoff estimates

Finally we must show

$$||\chi R(z,h)\chi||_{H_h^s \to H_h^{s+2}} \lesssim h^{-1}.$$

Lemma (Theorem 4.37, D-Z)

If χ is a cutoff such that $\chi V = V$ and $\chi P_\theta = \chi P$, and $\operatorname{Im} \sqrt{z} e^{i\theta} > 0$,

$$\chi (P - z)^{-1} \chi = \chi (P_\theta - z)^{-1} \chi.$$

Since

$$||(P_\theta - z)^{-1}||_{H_h^s \to H_h^{s+2}}$$

and we defined $\sqrt{\cdot}$ by $\operatorname{Im} \sqrt{z} > 0$, we can just take θ small enough that $\operatorname{Im} \sqrt{z} e^{i\theta} > 0$, and r_1 so large that $\chi P_\theta = \chi P$, to apply the lemma and conclude the claimed result.

Semiclassical defect measures

Lemma (Thm 5.2, SCA)

Suppose that $u(h)$ are functions, $||u(h)||_{L^2} = 1$. Let $a \in S^0(\mathbb{R}^n)$. Then there is a positive Radon measure $\mu \in C_{comp}(T^*\mathbb{R}^n)^*_+$ and a sequence $h_j \to 0$ such that

$$\lim_{j\to\infty} \langle a(h_j D)u(h_j), u(h_j)\rangle = \int_{T^*\mathbb{R}^n} a \, d\mu.$$

Definition

The measure μ is called the *semiclassical defect measure* that $u(h)$ converges to.

Example (Example 5.1.1, SCA)

If $u(h)$ is microlocalized to (x, ξ) then the unique semiclassical measure of u is $\delta_{(x,\xi)}$.

Semiclassical defect measures

Proof of lemma; quasimodes

To prove the existence of semiclassical defect measures, let $\{a_k\}_k \subset S^0$ be dense in $C_{comp}(T^*\mathbb{R}^n)$. By the Calderón-Vaillaincourt theorem and the Cantor–Arzelà–Ascoli diagonal argument, we can find $h_j \to 0$ such that

$$\int_{T^*\mathbb{R}^n} a_k \, d\mu = \lim_{j\to\infty} \langle a_k(h_j D)u(h_j), u(h_j)\rangle$$

exists and is $O(||a_k||_{L^\infty})$. By the Riesz-Markov theorem and the sharp Gårding inequality, μ is a positive Radon measure, which proves the lemma.

Definition

An ε-*quasimode* for a semiclassical pseudodifferential operator Q is a family of functions $u(h)$ with $||u(h)||_{L^2} = 1$ and $||Q(h)u(h)||_{L^2} < \varepsilon$.

By the lemma, every $o(h)$-quasimode converges to a (possibly nonunique) semiclassical defect measure.

Semiclassical defect measures

Defect measures of resonant states

Theorem (defect measures for resonant states)

Fix an energy region $0 < \alpha \leq E \leq \beta < \infty$ *and* $C_0 > 0$. *Let* $K = [\alpha, \beta] \times i[-C_0 h, C_0 h]$. *Suppose that* $z(h) \in K$ *and* $z(h) \to E$. *Let* $u(h)$ *be a* $o(h)$*-quasimode for the operator* $P_\theta(h) - z(h)$. *Choose* $h_j \to 0$ *such that* $\operatorname{Im} z(h_j)/h_j$ *converges, say to* ν, *and that* $u(h_j)$ *converges to a semiclassical defect measure* μ. *Then:*

- $\operatorname{supp} \mu \subseteq \Gamma_E^+$.
- *If* $U \supseteq K_E$ *is open, then* $\mu(U) > 0$.
- *If* $U \subseteq \{r \leq r_1\}$ *is open and* $t \geq 0$, *then*

$$\mu(e^{-tH_p}(U)) = e^{2\nu t} \mu(U).$$

Here the sequence h_j exists by compactness of $[-C_0 h, C_0 h]$ and the fact that the proof of the previous lemma allows us to restrict to a countable set of h's.

Semiclassical defect measures

Interpretation of theorem

Suppose that $E \in (\alpha, \beta)$.
Let $z_j \to E$ be a sequence of resonances of P and suppose that there are u_j such that

$$P_\theta(h_j)u_j = z_j u_j.$$

These u_j must exist, by general results about blackbox complex scaling, and we can think of them as perturbations of resonant states.
Passing to a subsequence we may assume that the u_j meet the hypotheses of the above theorem, so converge to a semiclassical defect measure μ. It follows that $K_{[\alpha,\beta]}$ is nonempty and hence P has trapping at the energy scale $[\alpha, \beta]$, since $\mu(K_{[\alpha,\beta]}) > 0$.
Thus this theorem is a partial converse to the previous theorem, which said that if P satisfied the nontrapping condition $K_{[\alpha,\beta]} = \emptyset$, then P had a resonance-free region at the energy scale $[\alpha, \beta]$.

Semiclassical defect measures

Proof of support properties

Lemma

$\operatorname{supp} \mu \subseteq \Gamma_E^+$.

Lemma (Thm 5.3, SCA)

Let q be a real symbol, let $u(h)$ be a $o(1)$-quasimode of $q(hD)$, and let μ be a semiclassical defect measure of u. Then $\operatorname{supp} \mu \subseteq q^{-1}(0)$.

We proved that $p_\theta^{-1}(E) \subseteq p^{-1}(E)$ so it follows that $\mu(p \neq E) = 0$.
If $a(h) \in C^\infty_{comp}(T^*\mathbb{R}^n)$ and $\operatorname{ess\,supp} a \cap \Gamma^+ = \emptyset$, then we proved that

$$||a(hD)u||_{L^2} \lesssim h^{-1}||(P_\theta(h) - z(h))u(h)||_{L^2} + h^\infty.$$

The right-hand side vanishes since $u(h)$ is a $o(h)$-quasimode of $P_\theta(h) - z(h)$, so

$$\int_{T^*\mathbb{R}^n} a \; d\mu = \lim_{h \to 0} \langle a(hD)u(h), u(h) \rangle = 0$$

so $\mu(T^*\mathbb{R}^n \setminus \Gamma^+) = 0$. This proves the lemma.

Semiclassical defect measures

Proof that trapped sets are nontrivial

Lemma

For every open $U \supseteq K_E$, $\mu(U) > 0$.

Let $b(h) \in C^\infty_{comp}(T^*\mathbb{R}^n)$ and suppose that $b(hD)$ is elliptic in a neighborhood of K_E and that $\operatorname{ess\,supp} b \subseteq U$. We proved the ellipticity estimates

$$||b(hD)u(h)||_{L^2} \gtrsim ||u(h)||_{L^2} - h^{-1}||(P_\theta(h) - z(h))u(h)||_{L^2} \gtrsim 1$$

uniformly in h. Taking the limit of $||b(hD)u(h)||_{L^2}^2 = \langle b(hD)^* b(hD)u(h), u(h)\rangle$, we conclude that

$$||b(h)||^2_{L^2(\mu)} = \int_{T^*\mathbb{R}^n} |b(h)|^2 \, d\mu \gtrsim 1.$$

But $b(h) = O(h^\infty)$ off U, so this is only possible if $\mu(U) > 0$. This proves the lemma.

Semiclassical defect measures

Proof of ergodic properties

Lemma

If $U \subseteq \{r \leq r_1\}$ is open, $t \geq 0$, $\operatorname{Im} z(h)/h \to \nu$, then $\mu(e^{-tH_p}(U)) = e^{2\nu t}\mu(U)$.

Let χ be a cutoff which neglects complex scaling, thus $\chi F_\theta = 0$ (so $\chi P_\theta = \chi P$). Since $u(h)$ is a $o(h)$-quasimode of $P_\theta - z$, it is also a $o(h)$-quasimode of $\chi(P_\theta - z)$.

Lemma (Thm E.44, D-Z)

Let $Q \in \Psi(\mathbb{R}^n)$, $q = \sigma_h(Q)$ real, and μ the semiclassical defect measure of a $o(h)$-quasimode of Q. Let $\operatorname{Im} Q = (Q - Q^*)/2i$ and $a \in C^\infty_{comp}(T^*\mathbb{R}^n)$; then

$$\int_{T^*M} H_q a \, d\mu = -2\langle a, \sigma_h(h^{-1} \operatorname{Im} Q)\rangle_{L^2(\mu)}.$$

This result generalizes Thm 5.4, SCA, which says that if q is a real symbol then the semiclassical defect measure of a $o(h)$-quasimode of $q(hD)$ is H_q-invariant.

Semiclassical defect measures

Proof of ergodic properties II

We apply the lemma with $q = \sigma_h(P(h) - \operatorname{Re} z(h) - i\nu h)$.
Here

$$||q(hD)u(h)||_{L^2} = ||\operatorname{Im} z(h) - i\nu h||_{L^2 \to L^2} + o(h) = o(h)$$

since $u(h)$ is a $o(h)$-quasimode of $P(h) - z(h)$ and $\operatorname{Im} z(h)/h \to \nu$.
Thus for every $a \in C^\infty_{comp}(B(0, r_1))$ (which is μ-almost preserved by H_p since H_p sends $\Gamma_E^+ \cap B(0, r_1)$ to itself, and μ is supported in Γ_E^+),

$$\int_{\Gamma_E^+} \frac{H_p}{2\nu} a \, d\mu = \int_{\Gamma_E^+} a \, d\mu.$$

But this means that

$$\int_{\Gamma_E^+} a \circ e^{tH_p} \, d\mu = e^{2\nu} \int_{\Gamma_E^+} a \, d\mu.$$

Taking $a \to 1_U$ for some U open we see the lemma and hence the theorem.